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Some Exact Solutions for Radiation View Factors from Spheres

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Based on a new formulation for radiative view factor from a sphere to a class of axisymmetric bodies, closed-form expressions are developed for the radiation view factors from a sphere to the external surface of a coaxial cylinder and from a sphere to an intersecting coaxial disk. The numerical results of the view factors are presented graphically in a wide range of parameters. At a limiting condition, one of the present formulas leads to the view factor from an annular ring to a coaxial hemisphere. The results are compared with those obtained earlier based on the Monte Carlo technique.

Nomenclature

dA_2	= differential element
$dF_{s\text{-}\delta r}$	= shape factor from a sphere to a coaxial differential right-circular ring
$dF_{s\text{-}\delta Ar}$	= shape factor from a sphere to a differential annular ring the plane of which intersects the sphere
$dF_{s\text{-}dA_2}$	= shape factor from a sphere to a differential element
$F_{s\text{-AD}}$	= shape factor from a sphere to a coaxial annular disk
$F_{s\text{-ID}}$	= shape factor from a sphere to a coaxial annular disk for which the inside radius is in contact with the sphere
$F_{s\text{-2ID}}$	= shape factor from a sphere to the both sides of an intersected coaxial disk
$F_{s\text{-cy}}$	= shape factor from a sphere to the external surface of a coaxial cylinder
h	= the radius of the coaxial cylinder or the distance from center of the sphere to the plane of the disk or the intersected disk
H	$= h/r$
ℓ_1	= distance between the center of the sphere to one end of the cylinder
ℓ_2	= distance between the center of the sphere to the other end of the cylinder
ℓ	$= \ell_2 - \ell_1$
L	$= \ell/r$
L_1	$= \ell_1/r$
L_2	$= \ell_2/r$
n_2	= vector normal to the surface of dA_2
p	$= \ell_1 - r$, distance between the center of the sphere and the cylinder
P	$= p/r$
r	= radius of the sphere
r_1	= internal radius of disk
r_2	= extenal radius of disk
R or R_2	$= r_2/r$
R_1	$= r_1/r$
s	= distance from the element dA_2 to the sphere
S	$= s/r$
θ	= angle between n_2 and the line connecting the element dA_2 and the center of the sphere
ξ	= angle between the normal vector n_2 and the y axis in Fig. 3

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Introduction

THE determination of radiation view factors between bodies is of great significance in the calculation of radiative heat exchange. However, very few exact solutions are available in the literature for the view factors between two finite surfaces, especially for three dimensional geometries. Analytical solutions for the configuration factors involving either a finite cylindrical, conical, or spherical body have been given in Refs. 1-5. Among them, Feingold and Gupta[3] developed closed-form solutions for the view factors from a sphere to a coaxial disk and a sphere to the internal surface of a larger coaxial cylinder. The results are of great interest in various engineering applications. However, the formulations have the following restrictions. 1) The configuration factor from a sphere to a coaxial disk is limited to the case when the disk and the sphere are not intersecting. 2) The view factor solution from a sphere to a cylinder is applicable only when the radius of the sphere is smaller than that of the cylinder. Their scheme does not yield analytical solutions for the view factors from a sphere to an intersected disk or ring and from a sphere to the external surface of a hollow cylinder or cylindrical rod. To the best knowledge of the present authors, the analytical solutions for the preceding view factors have not yet been available in the literature, although a numerical solution has been published for the radiation from annular ring to a coaxial hemisphere.[6] As a complement to Ref. 3, this paper presents two closed-form solutions for the aforementioned configuration factors using a similar approach introduced earlier by Chung and Naraghi[5] in which a new formulation was developed for the evaluation of view factor from a sphere to a class of coaxial axisymmetric bodies.

View Factor from a Sphere to the External Surface of a Coaxial Cylinder

We will start with the case of diffuse radiation from one side of a differential element to a sphere when the plane of this differential element intersects the sphere as shown in Fig. 1. The view factor for this configuration has been obtained earlier in Ref. 5 and is given by

$$F_{dA_2-s} = -\frac{\sqrt{(S+1)^2-1}\sin\theta}{\pi(S+1)^2}\sqrt{1-[(S+1)^2-1]\cot^2\theta} + \frac{1}{\pi}\tan^{-1}\left(\frac{\sin\theta\sqrt{1-[(S+1)^2-1]\cot^2\theta}}{\sqrt{(S+1)^2-1}}\right) + \frac{\cos\theta}{\pi(S+1)^2}\cos^{-1}\left(-\sqrt{(S+1)^2-1}\cot\theta\right) \quad (1)$$

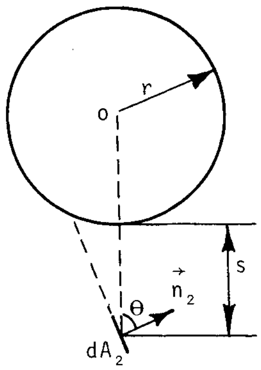

Fig. 1 Sphere-differential element view factor geometry with an arbitrary n_2.

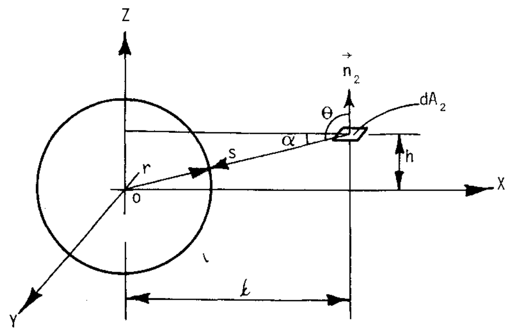

Fig. 2 Sphere-differential element view factor geometry with n_2 perpendicular to x-y plane.

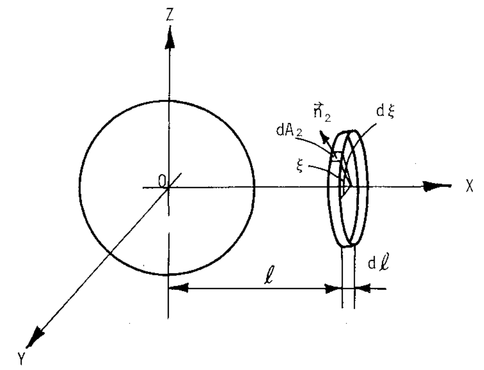

Fig. 3 Sphere-differential coaxial ring view factor geometry.

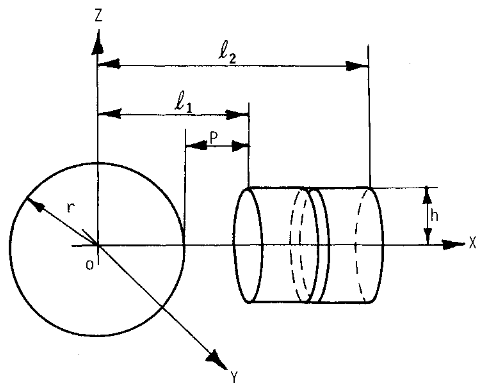

Fig. 4 Sphere-coaxial cylinder view factor geometry.

for

$$\cot^{-1}\frac{1}{\sqrt{(S+1)^2-1}}\leq\theta\leq\cot^{-1}\frac{-1}{\sqrt{(S+1)^2-1}}$$

where

$$S=s/r$$

We now consider a special case for which the normal of the radiating differential element n_2 is parallel to the z axis and lies in the x-z plane as shown in Fig. 2. It is seen from this figure that

$$\theta=(\pi/2)+\alpha\ =(\pi/2)+\tan^{-1}(H/L) \tag{2}$$

and also

$$S=\sqrt{H^2+L^2}-1 \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) yields

$$F_{\mathrm{d}A_2\text{-}s}=-\frac{\sqrt{(L^2+H^2-1)(1-H^2)}}{\pi(L^2+H^2)}$$

$$+\frac{1}{\pi}\tan^{-1}\sqrt{\frac{1-H^2}{L^2+H^2-1}}-\frac{H}{\pi(L^2+H^2)^{3/2}}$$

$$\times\cos^{-1}\frac{H\sqrt{L^2+H^2-1}}{L}\qquad -1\leq H\leq 1 \tag{4}$$

Let the differential element $\mathrm{d}A_2$ mentioned earlier rotate around the x axis and form a right circular differential ring with a radius h. The view factor from sphere to any differential element on this differential ring can be achieved from the reciprocity rule (see Fig. 3)

$$\mathrm{d}F_{s\text{-}\mathrm{d}A_2}=\frac{\mathrm{d}A_2}{4\pi r^2}F_{\mathrm{d}A_2\text{-}s}=\frac{h\mathrm{d}\xi\mathrm{d}\ell F_{\mathrm{d}A_2\text{-}s}}{4\pi r^2} \tag{5}$$

Integrating $\mathrm{d}F_{s\text{-}\mathrm{d}A_2}$ with respect to ξ and noting that $F_{\mathrm{d}A_2-s}$ is independent of the angle ξ due to symmetrical configuration, we have the view factor from the sphere to the coaxial differential ring.

$$\mathrm{d}F_{s\text{-}\delta r}=\int_0^{2\pi}\frac{h\mathrm{d}\ell}{4\pi r^2}F_{\mathrm{d}A_2\text{-}s}\mathrm{d}\xi=\frac{h}{2r^2}F_{\mathrm{d}A_2\text{-}s}\mathrm{d}\ell \tag{6}$$

Substituting Eq. (4) into Eq. (6) and integrating the resulting expression from $L_1=\ell_1/r$ to $L_2=\ell_2/r$ we obtain the following closed-form solution for the view factor from a sphere to the external surface of a coaxial hollow cylinder with a length $\ell_2-\ell_1$ and radius h (see Fig. 4):

$$F_{s\text{-cy}}=\frac{H}{2\pi}\left(L_2\tan^{-1}\sqrt{\frac{1-H^2}{L_2^2+H^2-1}}\right.$$

$$-L_1\tan^{-1}\sqrt{\frac{1-H^2}{L_1^2+H^2-1}}$$

$$-\frac{L_2}{H\sqrt{L_2^2+H^2}}\cos^{-1}\frac{H\sqrt{L_2^2+H^2-1}}{L_2}$$

$$\left.+\frac{L_1}{H\sqrt{L_1^2+H^2}}\cos^{-1}\frac{H\sqrt{L_1^2+H^2-1}}{L_1}\right) \tag{7}$$

The preceding formula is graphically illustrated in Figs. 5 and 6. Figure 5 shows the variation of $F_{s\text{-cy}}$ vs $H(=h/r)$ with $L_2-L_1=\ell/r$ as a parameter when the sphere and the hollow

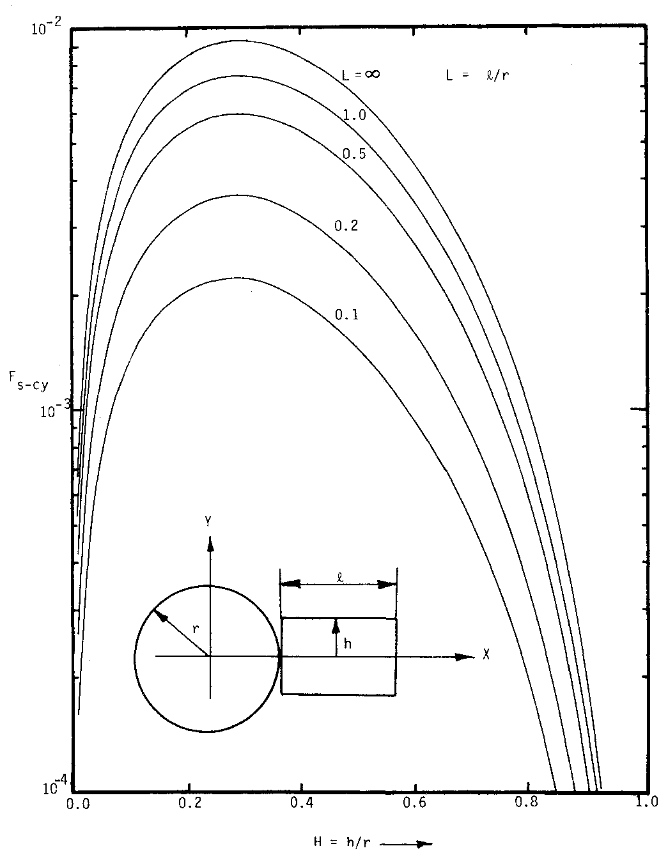

Fig. 5 Radiation view factors from a sphere to the external surface of a coaxial hollow cylinder with $p=0$.

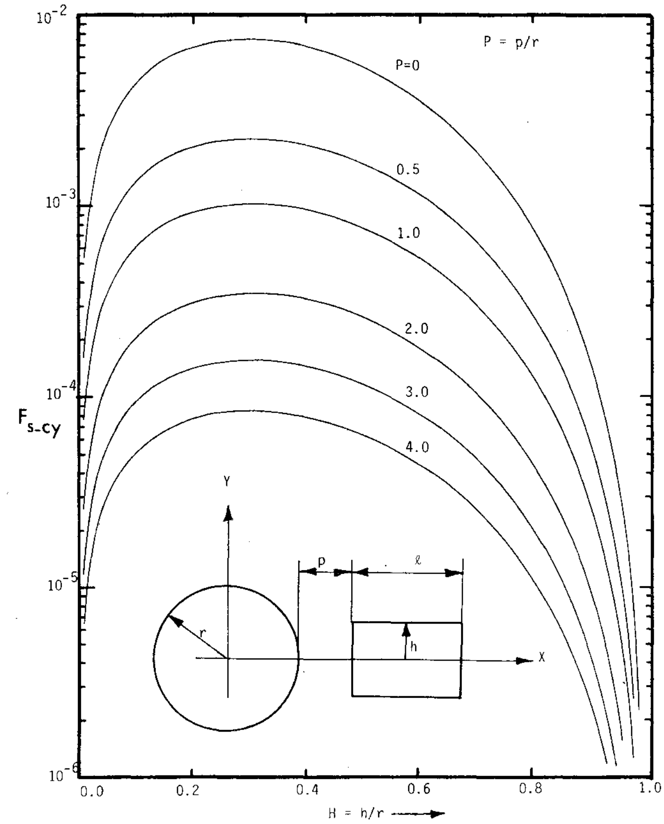

Fig. 6 Radiation view factors from a sphere to the external surface of a coaxial hollow cylinder with $\ell = r$.

cylinder are in contact. Figure 6 represents the changes of $F_{s\text{-cy}}$ with respect to H while P is a parameter and $\ell = r$. The view factor is zero when H is equal to both zero and unity. This implies that a maximum value of view factor exists at certain radius ratio. Interestingly, our numerical computations show that this maximum point is not very sensitive to the dimensions of P and L; it always locates in the neighborhood of $H=0.29$. With $F_{s\text{-cy}}$ available, other configuration factors, such as the shape factors from a sphere to the external surface of a sector of a hollow cylinder, the view factor from a sector of sphere to a coaxial hollow cylinder, etc., can be generated easily.

View Factors from a Sphere to an Intersecting Coaxial Disk

Consider again the configuration shown in Fig. 2. If we rotate the element dA_2 around the z axis we will have a coaxial ring around the sphere shown in Fig. 7. The shape factor from the sphere to the horizontal differential ring is given by

$$dF_{s\text{-}\delta Ar} = \int_0^{2\pi} \frac{\ell d\ell}{4\pi r^2} F_{dA_2\text{-}s} d\alpha$$

$$= \frac{\ell d\ell}{2r^2} F_{dA_2\text{-}s} \qquad (8)$$

Substituting Eq. (4) into Eq. (8) and integrating the result with respect to ℓ from r_1 to r_2 gives the shape factor from a sphere to the upper surface of an annular disk the plane of which intersects the sphere; and its internal and external radii are r_1 and r_2, respectively (see Fig. 7). Again the preceding single integral can be integrated exactly and is given by the following

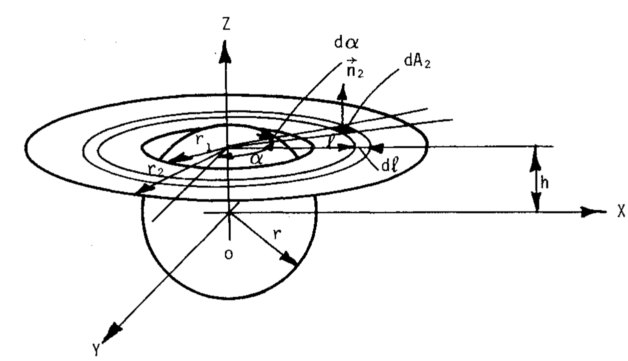

Fig. 7 Sphere-differential coaxial ring view factor geometry with n_2 perpendicular to x-y plane.

expression:

$$F_{s\text{-AD}} = \frac{1}{2\pi}\Bigg[\frac{\sqrt{1-H^2}}{2}\left(-\sqrt{R_2^2+H^2-1}+\sqrt{R_1^2+H^2-1}\right)$$

$$+\frac{R_2^2}{2}\tan^{-1}\sqrt{\frac{1-H^2}{R_2^2+H^2-1}}-\frac{R_1^2}{2}\tan^{-1}\sqrt{\frac{1-H^2}{R_1^2+H^2-1}}$$

$$+\frac{H}{\sqrt{R_2^2+H^2}}\cos^{-1}\frac{H\sqrt{R_2^2+H^2-1}}{R_2}$$

$$-\frac{H}{\sqrt{R_1^2+H^2}}\cos^{-1}\frac{H\sqrt{R_1^2+H^2-1}}{R_1}$$

$$+\tan^{-1}\sqrt{\frac{R_2^2+H^2-1}{1-H^2}}-\tan^{-1}\sqrt{\frac{R_1^2+H^2-1}{1-H^2}}\Bigg]$$

$$-1 \le H \le 1 \qquad (9)$$

When the inner circle of the annular disk is in contact with the sphere (see Fig. 8) we can eliminate R_1 in Eq. (9) from the relationship

$$R_1 = \sqrt{1-H^2}$$

There is obtained

$$F_{s\text{-}1D} = \frac{1}{2\pi}\left[-\frac{\sqrt{(1-H^2)(R^2+H^2-1)}}{2} + \frac{R^2}{2}\tan^{-1}\sqrt{\frac{1-H^2}{R^2+H^2-1}} - \frac{(1+2H-H^2)\pi}{4} + \frac{H}{\sqrt{R^2+H^2}}\cos^{-1}\frac{H\sqrt{R^2+H^2-1}}{R} + \tan^{-1}\sqrt{\frac{R^2+H^2-1}{1-H^2}}\right] \qquad -1 \le H \le 1 \tag{10}$$

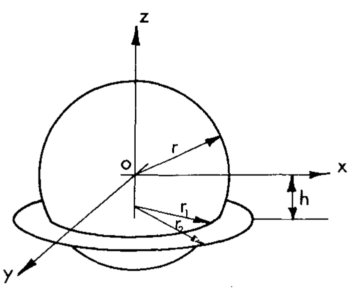

Fig. 8 Sphere-intersecting coaxial disk view factor geometry.

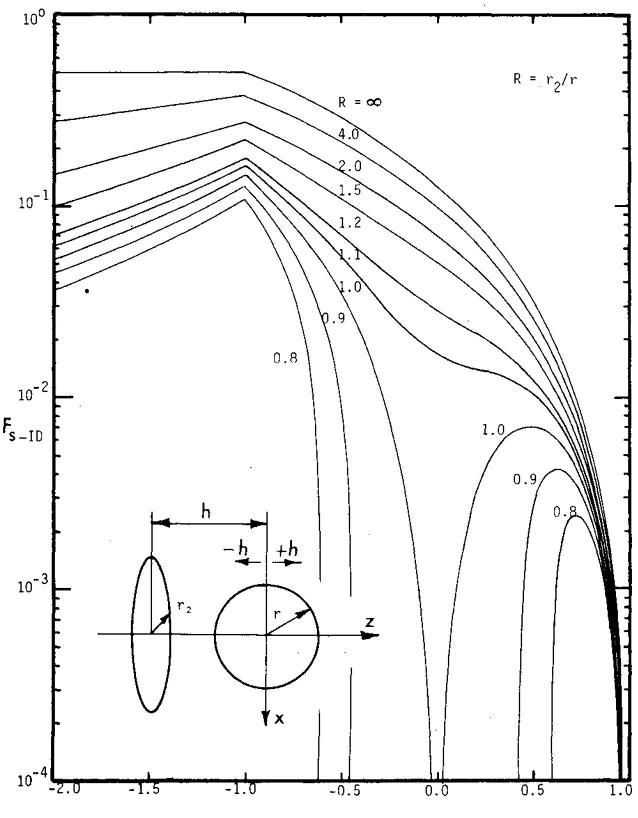

Fig. 9 Radiation view factors from a sphere to the right surface of a coaxial intersecting or unintersecting disk.

where R is the dimensionless outer radius of the intersected disk, r_2/r. Equations (9) and (10) represent the view factor from a sphere to the upper surface of the disk, shown in Fig. 8. Figure 9 shows the variations of the view factor when the distance between the center of the sphere and the center of the

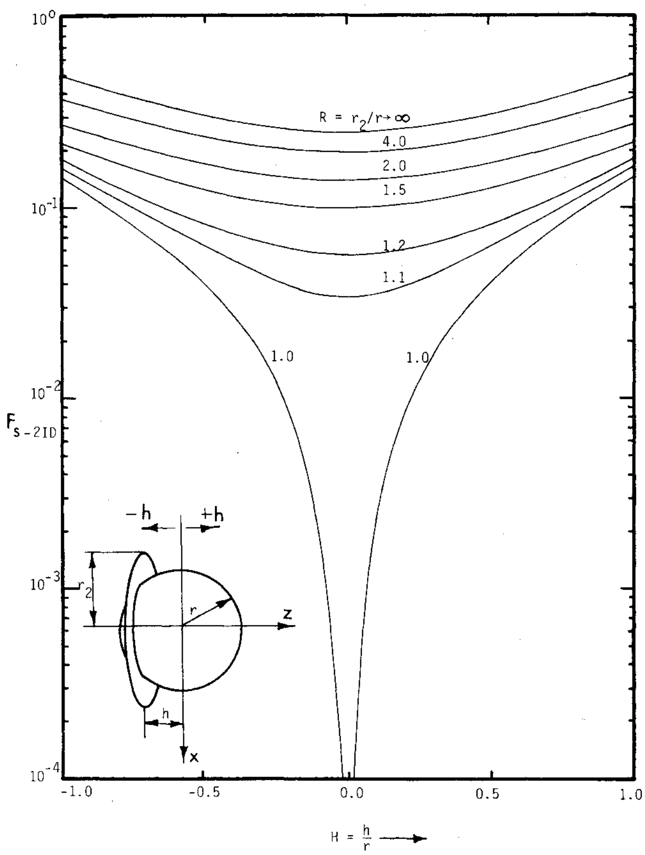

Fig. 10 Radiation view factors from a sphere to both sides of a coaxial intersecting disk.

Table 1 Radiation view factors from an annular ring to coaxial hemisphere

R	Present solution Eq. (11)	Monte Carlo solution Ref. 6
1.1	0.31993	0.300
1.2	0.25740	0.245
1.3	0.21593	0.209
1.4	0.18541	0.182
1.5	0.16175	0.160
1.6	0.14281	0.143
1.7	0.12728	0.126
1.8	0.11433	0.112
1.9	0.10337	0.103
2.0	0.09400	0.093
2.2	0.07886	0.076
2.4	0.06721	0.066
2.6	0.05803	0.059
2.8	0.05064	0.049
3.0	0.04461	0.045
3.2	0.03961	0.039
3.4	0.03542	0.035
3.6	0.03187	0.032
3.8	0.02883	0.027
4.0	0.02621	0.027
4.2	0.02394	0.024
4.4	0.02195	0.022
4.6	0.02020	0.022
4.8	0.01866	0.019
5.0	0.01728	0.017

disk varies from -2 to 1 at different values of R. In the sketch of Fig. 9, positive H is in the positive direction of z coordinate. When H is less than -1, the disk does not intersect the sphere; the solution of Feingold and Gupta[3] is thus employed in this region. It is seen from Fig. 9 that there is a maximum value of view factor at $H=-1$ when the sphere is in contact with the disk. The view factor decreases as H increases. It is of interest to examine the special case with $R=1$. The view factor drops rapidly as the disk moves from $H=-1$ toward the right direction and reaches a minimum of zero at $H=0$, since the disk disappears at the center of the sphere $(\because r_2=r)$. The area fraction of the disk outside the sphere increases as it continues to move toward the right. The view factor starts to increase and then decrease to zero again at $H=1$, since at this position, the sphere can not "see" the right-hand side of the disk at all. As can be seen from Fig. 9, an upper bound of the view factor, $F_{s\text{-}1D}=0.5$ exists at $H=-1$ when $R\to\infty$. This corresponds to the situation of radiation from sphere to an infinite wall.

Another interesting limiting case is the configuration factor from an annular ring to a coaxial hemisphere; a case has been studied by Ballance and Donovan[6] using the Monte Carlo method. The solution can be derived directly from Eq. (10). Note that the view factor from a sphere to a coaxial intersecting disk is identical to that from the sphere to the annular ring and the view factor from one side of an annular ring to a coaxial sphere is identical to that from the same surface to the hemisphere when the disk cuts the sphere at the centerline. Therefore, setting $H=0$ in Eq. (10) and applying the reciprocity rule, we obtain the view factor from an annular ring to a coaxial hemisphere of the form

$$F_{1D\text{-}s}=\frac{1}{(R^2-1)\pi}\left(-\frac{\pi}{2}-\sqrt{R^2-1}+R^2\tan^{-1}\frac{1}{\sqrt{R^2-1}}+2\tan^{-1}\sqrt{R^2-1}\right) \tag{11}$$

Table 1 shows the comparison between the present solution and the Monte Carlo solution with a sample size of 60,000 paths from the annular to the hemisphere. It appears that 60,000 paths is not enough in order to obtain accurate results within an error of ±2% claimed by the authors.

To determine the view factor from the sphere to the other side of the disk we simply replace H by $-H$ in Eqs. (9) and (10). If the view factor from the sphere to both sides of the intersecting disk is sought, we merely take the sum of $F_{s\text{-}1D}$ based on both H and $-H$. The final result is

$$F_{s\text{-}21D}=\frac{1}{2\pi}\left[-\sqrt{(1-H^2)(R^2+H^2-1)}+R^2\tan^{-1}\sqrt{\frac{1-H^2}{R^2+H^2-1}}-\frac{(1-H^2)\pi}{2}-\frac{2H}{\sqrt{R^2+H^2}}\left(\frac{\pi}{2}-\cos^{-1}\frac{H\sqrt{R^2+H^2-1}}{R}\right)+2\tan^{-1}\sqrt{\frac{R^2+H^2-1}{1-H^2}}\right] \tag{12}$$

Note that $F_{s\text{-}21D}$ is not equal to twice $F_{s\text{-}1D}$ since $F_{s\text{-}1D}$ in Eq. (10) is not completely symmetrical with respect to H. However, $F_{s\text{-}21D}$ is symmetrical with respect to H as can be seen from Eq. (12). Figure 10 depicts $F_{s\text{-}21D}$ as a function of H with R as a parameter. As expected at $H=0$ a minimum value of $F_{s\text{-}21D}$ always exists on each curve.

Conclusion

Complementary to Ref. 3, the present paper provides additional analytical solutions for the radiation view factors from a sphere. The exact solutions would have been extremely difficult and tedious if the conventional quadruple integration technique were employed.

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